What Happens to the Schwarzschild Solution in Quantum Corrected Gravity?

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K.S.S., Gen.Rel.Grav. 9 (1978) 353
W. Nelson, Phys.Rev. D82 (2010) 104026, arXiv:1010.3986
H. Lü, A. Perkins, C.N. Pope & K.S.S., to appear

Quantum Context

One-loop quantum corrections to General relativity in 4-dimensional spacetime produce ultraviolet divergences of curvature-squared structure.

G. 't Hooft and M. Veltman, *Ann. Inst. Henri Poincaré* **20**, 69 (1974)

Inclusion of $\int d^4x\sqrt{-g}(\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \rho R^2)$ terms ab initio in the gravitational action leads to a renormalizable $D = 4$ theory, but at the price of a loss of *unitarity* owing to the modes arising from the $\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ term, where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor.

K.S.S., *Phys. Rev.* **D16**, 953 (1977).

[In $D = 4$ spacetime dimensions, this $(\mathrm{Weyl})^2$ term is equivalent, up to a topological total derivative *via* the Gauss-Bonnet theorem, to the combination $\alpha(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2)$].

Despite the apparent nonphysical behavior, quadratic-curvature gravities continue to be explored in a number of contexts:

- The *asymptotic safety scenario* considers the possibility that there may be a non-Gaussian renormalization-group fixed point and associated flow trajectories on which the ghost states arising from the $(\text{Weyl})^2$ term could be absent.

  S. Weinberg 1976, M. Reuter 1996, M. Niedermaier 2009

- *Cosmology:* Starobinsky's original model for inflation was based on a $\int d^4x\sqrt{-g}(-R+\rho R^2)$ model.

  A.A. Starobinsky 1980; Mukhanov & Chibisov 1981

  Strikingly, this early model turns out to give the best current fit to CMB fluctuation data from the Planck satellite.

  J. Martin, C. Ringeval and V. Vennin, arXiv:1303.3787

Classical gravity with higher derivatives

We shall not try here to settle philosophical debates about various attitudes that can be taken towards the implementation of quantum corrections (Wilsonian, or other), but shall simply adopt a point of view taking the higher-derivative terms and their consequences for gravitational field-theory solutions seriously.

Accordingly, we shall consider the gravitational action $I = -\int d^4x\sqrt{-g}(\alpha R_{\mu\nu}R^{\mu\nu} - \beta R^2 + \gamma\kappa^{-2}R)$, which can also be rewritten $I = -\int d^4x\sqrt{-g}(\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + (\frac{\alpha}{3} - \beta)R^2 + \gamma\kappa^{-2}R)$, so in terms of the earlier parametrization one has $\rho = \frac{\alpha}{3} - \beta$.

The field equations following from this higher-derivative action are

$$\begin{aligned} H_{\mu\nu} \;=\; & (\alpha - 2\beta)\nabla_\nu\nabla_\mu R - \alpha\nabla^\eta\nabla_\eta R_{\mu\nu} - (\frac{\alpha}{2} - 2\beta)g_{\mu\nu}\nabla^\eta\nabla_\eta R \\ & +2\alpha R^{\eta\lambda}R_{\mu\eta\nu\lambda} - 2\beta R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(\alpha R^{\eta\lambda}R_{\eta\lambda} - \beta R^2) \\ & +\gamma\kappa^{-2}R_{\mu\nu} - \frac{1}{2}\gamma\kappa^{-2}g_{\mu\nu}R = T_{\mu\nu} \end{aligned}$$

Separation of modes in the linearized theory

Solving the full nonlinear field equations is clearly a challenge. One can make initial progress by restricting the metric to infinitesimal fluctuations about flat space, defining $h_{\mu\nu} = \kappa^{-1}(g_{\mu\nu} - \eta_{\mu\nu})$ and restricting attention to field equations linearized in $h_{\mu\nu}$, or equivalently by restricting attention to quadratic terms in $h_{\mu\nu}$ in the action.

The action then becomes

$$\begin{aligned} I_{\rm Lin} &= \int d^4x \{ -\frac{1}{4} h^{\mu\nu}(\alpha\kappa^2\Box - \gamma)\Box P^{(2)}_{\mu\nu\rho\sigma} h^{\rho\sigma} \\ &\quad + \frac{1}{2} h^{\mu\nu}[2(3\beta - \alpha)\kappa^2\Box - \gamma]\Box P^{(0;\mathrm{s})}_{\mu\nu\rho\sigma} h^{\rho\sigma} \} \ ; \end{aligned}$$

$$\begin{aligned} P^{(2)}_{\mu\nu\rho\sigma} &= \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - P^{(0;\mathrm{s})}_{\mu\nu\rho\sigma} \\ P^{(0;\mathrm{s})}_{\mu\nu\rho\sigma} &= \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \qquad \theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu} \qquad \omega_{\mu\nu} = \partial_\mu\partial_\nu/\Box \ , \end{aligned}$$

where the indices are lowered and raised with the background metric $\eta_{\mu\nu}$.

From this linearized action one deduces the dynamical content of the linearized theory: *positive-energy* massless spin-two, *negative-energy* massive spin-two with mass $m_2 = \gamma^{\frac{1}{2}}(\alpha\kappa^2)^{-\frac{1}{2}}$ and *positive-energy* massive spin-zero with mass $m_0 = \gamma^{\frac{1}{2}}[2(3\beta - \alpha)\kappa^2)]^{-\frac{1}{2}}$. K.S.S. 1978

$\sim\sim\sim$

A simple model of what has happened can be made with a single scalar field and a higher-derivative action coupled to a source J:

$$I_{\rm hd} = \int d^4x(-\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \tfrac{1}{2}\alpha\partial_\mu\phi\Box\partial^\mu\phi + J\phi)$$

Going over to momentum space k^μ, one can solve for ϕ and then separate the propagator into partial fractions:

$$\phi = \frac{J/\alpha}{k^2(k^2 + 1/\alpha)} = \frac{J}{k^2} - \frac{J}{k^2 + 1/\alpha}$$

similar to the structure found in quadratic gravity, but without the spin complications.

Static and spherically symmetric solutions

Now we come to the question of what happens to spherically symmetric gravitational solutions in the higher-curvature theory. One may choose to work in traditional Schwarzschild coordinates, for which the metric is given by

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

In the linearized theory, one then finds the general solution to the source-free field equations $H^L_{\mu\nu} = 0$, where $C, C^{2,0}, C^{2,+}, C^{2,-}, C^{0,+}, C^{0,-}$ are integration constants:

$$A(r) =$$

$$1 - \frac{C^{20}}{r} - C^{2+}\frac{e^{m_2 r}}{2r} - C^{2-}\frac{e^{-m_2 r}}{2r} + C^{0+}\frac{e^{m_0 r}}{r} + C^{0-}\frac{e^{-m_0 r}}{r}$$
$$+\frac{1}{2}C^{2+}m_2 e^{m_2 r} - \frac{1}{2}C^{2-}m_2 e^{-m_2 r} - C^{0+}m_0 e^{m_0 r} + C^{0-}m_0 e^{-m_0 r}$$

$$B(r) =$$

$$C + \frac{C^{20}}{r} + C^{2+}\frac{e^{m_2 r}}{r} + C^{2-}\frac{e^{-m_2 r}}{r} + C^{0+}\frac{e^{m_r}}{r} + C^{0-}\frac{e^{-m_0 r}}{r}$$

- As one might expect from the dynamics of the linearized theory, the general static, spherically symmetric solution is a combination of a massless Newtonian $1/r$ potential plus rising and falling Yukawa potentials arising in both the spin-two and spin-zero sectors.
- When coupling to non-gravitational matter fields is made *via* standard $h^{\mu\nu}T_{\mu\nu}$ minimal coupling, one gets values for the integration constants from the specific form of the source stress tensor. Requiring asymptotic flatness and coupling to a point-source positive-energy matter delta function $T_{\mu\nu} = \delta^0_\mu \delta^0_\nu M \delta^3(\vec{x})$, for example, one finds

$$
\begin{aligned}
A(r) &= 1 + \frac{\kappa^2 M}{8\pi\gamma r} - \frac{\kappa^2 M(1+m_2 r)}{12\pi\gamma}\frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M(1+m_0 r)}{48\pi\gamma}\frac{e^{-m_0 r}}{r} \\
B(r) &= 1 - \frac{\kappa^2 M}{8\pi\gamma r} + \frac{\kappa^2 M}{6\pi\gamma}\frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M}{24\pi\gamma}\frac{e^{-m_0 r}}{r}
\end{aligned}
$$

with specific combinations of the Newtonian $1/r$ and falling Yukawa potential corrections arising from the spin-two and spin-zero sectors.

What about the Schwarzschild solution?

Returning to the full nonlinear field equations in the source-free case $H_{\mu\nu} = 0$, one notes directly that any solution to the source-free Einstein equation $R_{\mu\nu} = 0$ will also be a solution to the higher-curvature theory's *source-free* equations. But do we really want such solutions now?

In the above toy scalar higher-derivative model, the source-free field equation is $(\Box - m^2)\Box\phi = 0$. While it is true that any genuine solution to $\Box\phi = 0$ satisfies the source-free higher-derivative equations, things go wrong when one considers the standard q/r solution to the *sourced* static problem $\nabla^2\phi = q\delta^3(\vec{x})$.

In order for this to be a solution to the higher-derivative theory, the source on the right-hand side of the field equation would need to be of the form $q(\nabla^2 - m^2)\delta^3(\vec{x})$. This is a highly singular distribution, and is *not even positive* when integrated with a generic profile function. In other words, the attempt to claim solutions to the second-order $\Box\phi = 0$ theory as solutions for the higher-derivative theory implies couplings to other "matter" fields without energy positivity.

From the above discussion, we conclude that, although the Schwarzschild solution is an apparent solution to the source-free higher-derivative equations $H_{\mu\nu} = 0$, it will not be a good solution arising from normal minimal coupling of gravity to matter fields. The sought-for solution should, in the weak-field linearized limit, display Yukawa corrections to the Newtonian $1/r$ potential at spatial infinity.

Now consider the full nonlinear field equations for the spherically symmetric case, once again source-free. They are somewhat frightful. Initially, one gets one third-order equation and one fourth-order equation. However, the system can then be rearranged into a system with two third-order equations for the two metric variables $A(r)$ and $B(r)$.

The first equation contains the third-order derivative $B^{(3)} = B'''$:

$$
\begin{aligned}
0 \;=\; & \frac{\sqrt{AB}}{16r^2A^5B^4}\Big(16(A-1)A^2B^4(5\alpha+(\alpha-2\beta)A-14\beta)-32r(\alpha-4\beta)A^2B^3B' \\
& +4r^2B^2\Big(7(3\alpha-8\beta)B^2A'^2+2AB\left(2(8\beta-3\alpha)BA''+(16\beta-5\alpha)A'B'\right) \\
& +A^2\left(8(\alpha-4\beta)BB''+(32\beta-11\alpha)B'^2\right)+4\gamma A^4B^2-4\gamma A^3B^2\Big) \\
& -4r^3B\Big(4ABB''\left((\alpha-4\beta)BA'+(\alpha-6\beta)AB'\right) \\
& +B'\left(-7(\alpha-4\beta)B^2A'^2+4AB\left((\alpha-4\beta)BA''+\beta A'B'\right)-(\alpha-8\beta)A^2B'^2+4\gamma A^3B^2\right)\Big) \\
& +r^4(\alpha-2\beta)\Big(6ABB'^2\left(A'B'-2AB''\right) \\
& +B^2\left(-8AA'B'B''+B'^2\left(7A'^2-4AA''\right)-4A^2B''^2\right)+7A^2B'^4\Big) \\
& +B^{(3)}\left(8r^4(\alpha-2\beta)A^2B^2B'+16r^3(\alpha-4\beta)A^2B^3\right)\Big)
\end{aligned}
$$

The second equation contains the third-order derivative $A^{(3)} = A'''$:

$$
\begin{aligned}
0 \;=\; & \frac{\sqrt{AB}}{2r^2A^5B^4\left(2(\alpha-4\beta)B+r(\alpha-2\beta)B'\right)^2}\Bigg(\\
& 16(\alpha-3\beta)(\alpha-4\beta)(A-1)A^3(\alpha+2\beta+(\alpha-2\beta)A)B^5 \\
& +16(\alpha-3\beta)A^2B^4\Big(2(\alpha-4\beta)(-2\alpha+2\beta+(\alpha-2\beta)A)BA' \\
& \quad +(\alpha-2\beta)(A-1)A(\alpha+2\beta+(\alpha-2\beta)A)B'\Big)r \\
& +4AB^3\Big(-4\beta(\alpha-4\beta)\gamma B^2A^4-4\left(-\beta(\alpha-4\beta)\gamma B^2+(\alpha-2\beta)^2(\alpha-3\beta)B'^2\right)A^3 \\
& \quad +(\alpha-3\beta)B'\left(4BA'(\alpha-2\beta)^2+\left(3\alpha^2-4\beta\alpha-16\beta^2\right)B'\right)A^2 \\
& \quad -2(\alpha-3\beta)B\left(\left(3\alpha^2-8\beta\alpha+8\beta^2\right)A'B'-2\alpha(\alpha-4\beta)BA''\right)A \\
& \quad -(\alpha-4\beta)(\alpha-3\beta)(5\alpha+8\beta)B^2A'^2\Big)r^2 \\
& +4(\alpha-4\beta)B^2\Big(+2(\alpha-2\beta)\gamma B^2B'A^5+2(\alpha-2\beta)\gamma B^2B'A^4 \\
& \quad +\left(-4\beta\gamma A'B^3+2(\alpha+4\beta)(\alpha-3\beta)B'B''B-(\alpha-3\beta)(3\alpha+4\beta)B'^3\right)A^3 \\
& \quad +2(\alpha-3\beta)B\left(\alpha BB'A''+A'\left(\alpha BB''-2(\alpha+\beta)B'^2\right)\right)A^2 \\
& \quad -\alpha(\alpha-3\beta)B^2A'\left(5A'B'+26BA''\right)A+28\alpha(\alpha-3\beta)B^3A'^3\Big)r^3 \\
& +(\alpha-2\beta)B\Big(-4\gamma B^2\left((\alpha-6\beta)B'^2-2(\alpha-4\beta)BB''\right)A^4 \\
& \quad +\Big(-(\alpha-3\beta)(5\alpha+4\beta)B'^4+8\alpha(\alpha-3\beta)BB''B'^2-4(\alpha-2\beta)\gamma B^3A'B' \\
& \quad -4(\alpha-4\beta)(\alpha-3\beta)B^2B''^2\Big)A^3 \\
& \quad +2(\alpha-3\beta)BB'\left(4\alpha BB'A''+A'\left((4\beta-5\alpha)B'^2+2(3\alpha-8\beta)BB''\right)\right)A^2 \\
& \quad -(\alpha-3\beta)B^2A'B'\left(3(7\alpha-4\beta)A'B'+52\alpha BA''\right)A+56\alpha(\alpha-3\beta)B^3A'^3B'\Big)r^4 \\
& -(\alpha-2\beta)^2AB'\left(AB'^2+B\left(A'B'-2AB''\right)\right)\Big(+2\gamma A^2B^2+(\alpha-3\beta)A'B'B \\
& \quad +(\alpha-3\beta)A\left(B'^2-2BB''\right)\Big)r^5 \\
& +\left(16r^3\alpha(\alpha-3\beta)(\alpha-4\beta)A^2B^5+8r^4\alpha(\alpha-2\beta)(\alpha-3\beta)A^2B'B^4\right)A^{(3)}\Bigg)
\end{aligned}
$$

An Israel Theorem

Hopes for an analytic solution to the static spherically symmetric equations are clearly rather slim. In the end, it will be necessary to explore such solutions by numerical means. However, some definite conclusions can be reached by analytical methods. A key tool in this analysis is an extension of Werner Israel's "no-hair" theorem.

W. Nelson, Phys.Rev. D82 (2010) 104026; arXiv:1010.3986; H. Lü, A. Perkins, C.N. Pope & K.S.S. to appear.

This theorem extends the classic Israel-Lichnerowicz theorem of GR to the Einstein-plus-quadratic-curvature gravity theories for static and spherically symmetric solutions. The approach is a standard one for "no-hair" theorems: find an appropriate tensorial factor to contract with the $H_{\mu\nu}$ field equations and then integrate out from a presumed horizon null-surface to asymptotically flat infinity. Provided that contraction with the right tensorial factor has been made, integration by parts then yields an integrand formed from a sum of squares all with the same sign, plus boundary terms and one more type of term that will have the same sign as the sum of squares provided two inequalities are respected.

Some flavor of how this Israel theorem is derived:

- For $3\beta - \alpha > 0$ (*i.e.* for non-tachyonic $m_0^2 > 0$), take the trace of the $H_{\mu\nu} = 0$ field equation: $\left(\Box - \frac{\gamma\kappa^{-2}}{2(3\beta-\alpha)}\right) R = 0$. Then multiply by $\lambda^{\frac{1}{2}} R$ and integrate with $\int \sqrt{h}$ over a spatial slice at a fixed time, where h_{ab} is the spatial metric and $\lambda = -t^a t^b g_{ab}$ is the norm of the timelike Killing vector t^a orthogonal to the spatial slice. Integrating by parts, one obtains

$$\int d^3x \sqrt{h} [D^a(\lambda^{\frac{1}{2}} R D_a R) - \lambda^{\frac{1}{2}} (D^a R)(D_a R) - m_0^2 \lambda^{\frac{1}{2}} R^2] = 0$$

  where D_a is a 3D covariant derivative on the spatial slice.

  From this, provided the boundary term arising from the total derivative gives a zero contribution, and for $m_0^2 > 0$, one obtains $R = 0$. The boundary at spatial infinity gives a vanishing contribution provided $R \to 0$ as $r \to \infty$.

- The inner boundary at a horizon null-surface will give a zero contribution since $\lambda = 0$ there.

In the general non-tachyonic case for $\alpha > 0$ and $3\beta - \alpha > 0$, one needs to complete the discussion using the non-trace part of the field equation. The derivation then goes similarly, with a surface term that vanishes on a null-surface and at asymptotically flat spatial infinity. One again obtains a requirement for the vanishing of an integral over the spatial slice of a sum of squares with the same (negative) sign, plus two final terms that are also of the same sign provided certain inequalities are obeyed.

From the required vanishing of this integral, one finds that, provided the following inequalities are satisfied

$$m_2^2 - {}^{(3)}R \geq 0$$
$$m_2^2 \bar{R}^a{}_b \bar{R}^b{}_a + 2\bar{R}^a{}_b \bar{R}^b{}_c \bar{R}^c{}_a \geq 0$$

one must have $\bar{R}_{ab} = 0$ and ${}^{(3)}R = 0$, where $\bar{R}_{ab}$ is the pull-back of the $D = 4$ Ricci tensor to the $D = 3$ spatial slice. Together, these imply $R_{\mu\nu} = 0$, requiring the solution to be Schwarzschild.

Implications of the Israel Theorem

The Israel theorem is a quite strong result, clearly incompatible with the corrections to the Schwarzschild solution found in the linearized analysis. Yet, for any spherically symmetric solution that can be continuously parametrically deformed to flat space, the linearized theory solutions must inevitably become relevant (as they do for the Schwarzschild solution itself, in the limit of vanishing Schwarzschild mass M).

- As we have seen , the Schwarzschild solution is *not* the solution that couples normally to a positive-energy stress tensor source. Solutions that couple to matter *via* minimal coupling ($\eta_{\mu\nu} \to g_{\mu\nu}$ with corresponding covariant derivatives ∇_μ) necessarily acquire corrections on top of the usual GR solution.

So the conclusion must be that one of the assumptions of the Israel theorem has to be violated in order to obtain an asymptotically-flat spherically-symmetric solution that can couple normally to non-gravitational matter.

- For asymptotically-flat solutions, the only other assumption is that there is a horizon.
- Thus one concludes that solutions arising from minimal coupling to non-gravitational fields have a strikingly different structure to those of ordinary Einstein theory: *there is no horizon* – black holes are not so black.
- The question then remains what the properly coupling solutions are actually like: what is the near-origin structure of spherically-symmetric and asymptotically flat solutions to the higher-derivative theory?

Indicial Analysis

A type of asymptotic analysis of the field equations complementary to the linearized analysis at $r \to \infty$ spatial infinity is study of the *indicial equations* for behavior as $r \to 0$. K.S.S. 1978 Let

$$\begin{aligned} A(r) &= a_s r^s + a_{s+1} r^{s+1} + a_{s+2} r^{s+2} + \cdots \\ B(r) &= b_t r^t + b_{t+1} r^{t+1} + b_{t+2} r^{t+2} + \cdots \end{aligned}$$

and analyze the conditions necessary for the lowest-order terms in r of the field equations $H_{\mu\nu} = 0$ to be satisfied. This gives the following results, for the general α, β theory:

$$\begin{aligned} (s,t) &= (1,-1) \qquad \text{with 3 free parameters} \\ (s,t) &= (0,0) \qquad \text{with 4 free parameters} \\ (s,t) &= (2,2) \qquad \text{with 6 free parameters} \end{aligned}$$

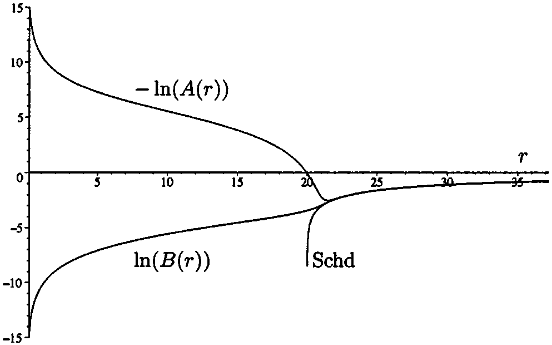
However, for the $(1,-1)$ and $(0,0)$ cases, the Israel theorem can once again be used to rule out these cases as candidates for solutions that match to the Yukawa-corrected asymptotically-flat solutions at infinity. This leaves the $(2,2)$ behavior at the origin as the unique remaining candidate for such solutions.

Numerical Analysis

In the absence of a suitably general analytic solution to the higher-derivative equations $H_{\mu\nu} = 0$, one must have recourse to numerical studies. This has been investigated by Bob Holdom.

B. Holdom, Phys.Rev. D66 (2002) 084010; hep-th/0206219

Here is a graph of his results, showing, indeed, r^2 behavior for both $A(r)$ and $B(r)$ as $r \to 0$, but connecting on to a Yukawa-corrected approximation to the Schwarzschild solution as $r \to \infty$:



Taking this numerical study together with the implications of the Israel theorem, a coherent picture emerges:

- The link between behavior near the origin, $r \to 0$, to asymptotically-flat Yukawa-corrected solutions at infinity happens in the $(s, t) = (2, 2)$ class of solutions to the higher-derivative theory. Note that the number of free parameters at the origin for this class matches precisely the number of parameters in the linearized solution. (Of course, rising Yukawa terms need to be excluded from the asymptotically flat solution set, but they are still solutions to the linearized theory.)
- *There is no horizon* in this set of minimally-coupled, Yukawa-corrected solutions. Solutions asymptotically approach the Schwarzschild solution for large r, but differ strikingly in what would have been the inner-horizon region.

- Although there is a curvature singularity at the origin in the $(2, 2)$ class of solutions (*e.g.* for this class, one has $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 20a_2^{-2}r^{-8} + \cdots$), this is a *timelike* singularity, unlike the *spacelike* singularity of the Schwarzschild solution.

Although one might complain that this non-Schwarzschild behavior occurs in a theory with a massive spin-two ghost, the limit as $\alpha \to 0$ removes this ghost as well as the complications of the m_2-dependent inequalities. The $R + R^2$ theory at $\alpha = 0$ is ghost-free, and yet has the same horizonless structure for its spherically symmetric static solutions as in the general α, β case, when its spherically-symmetric solution is derived from minimal coupling to non-gravitational matter.